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Quantum Theory and the Category of Complex Numbers

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The problem of developing a formalism of quantum theory, which is both consistent with the reality of the measurements and with the invariance properties of relativistic theories, is considered. A solution is found by using a real formulation of quantum mechanics, such that there exists an interpretation of the real properties of a physical system at all times. It is demonstrated also that several concepts in quantum field theory can be recast in a real formalism.

KEY WORDS: quantum; complex; imaginary unit; real formalism; S-matrix.

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1. INTRODUCTION

The introduction of an additional structure which distinguishes between the complex numbers from the real number system and the physical nature of complex quantities does not directly follow from the real dimensions of physical observation. The definition of the complex numbers with the introduction of the imaginary unit i such that $i^2 = -1$ or the product of two-component real vectors $(x_1, y_1) \cdot (x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1)$ is based on a condition that cannot be satisfied by any real number or equivalently a vector with a single component, which would describe the physical measurement of one dimension. This complex formalism is only consistent with physical theories when the measurements require a specification of several data points forming the component of a vector. An example is the introduction of complex numbers derived from the Lorentzian structure of space-time through the isomorphism $\overline{SO}(3, 1, \uparrow) = SL(2, C)$.

Although it may be convenient to use $SL(2, \mathbb{C})$, the group of orthochronous transformations $SO(3, 1, \uparrow)$ is sufficient for the description of Lorentz transformations of four-vectors in classical relativitic theories. The 2-1 mapping between the two groups, where two matrices $\Lambda^A{}_B = [\exp(-\frac{1}{2}\omega_{ab}\Sigma^{ab})]^A_B$, $(\Sigma^{ab})^A_B =$

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$-\sigma^{[a|AA'|}\sigma^{b]}{}_{BA'}$, and $-\Lambda^{A}{}_{B}$ correspond to a single Lorentz transformation $\Lambda^{\mu}{}_{\nu} = [\exp(-\frac{1}{2}\omega_{ab}\Sigma^{ab})]^{\mu}{}_{\nu}$, $(\Sigma^{ab})^{\mu}{}_{\nu} = -\frac{1}{2}[\gamma^{a}, \gamma^{b}]^{\mu}{}_{\nu}$, is indicative of the introduction of spinors. With two-component spinors, particles and anti-particles are distinguished by unprimed and primed spinors. The transformation $\Lambda^{A}{}_{B} \rightarrow \Lambda^{A'}{}_{B'}$, which interchanges positive and negative energy states and therefore particles and anti-particles, also can be interpreted as a reversal of the direction of the particle's propagation. Enlargement of the transformation group from $SO(3, 1, \uparrow)$ to $SO(3, 1, \uparrow) \cup SO(3, 1, \downarrow)$, can be implemented with the matrix

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

It is noted in Section 3 that there is a real basis for the Dirac gamma matrices. Since the gamma matrices are constructed from van der Waerden symbols, there exists a set of real matrices which can represent the Lorentz transformation on spinors and vectors.

The imaginary unit appears to be essential to a description of quantum mechanics based on wavefunctions. This feature leads to the Copenhagen interpretation of quantum mechanics for consistency with the reality of measurements. Nevertheless, a complete fomulation of quantum theory would require an intepretation of the measurements which is invariant under known symmetries such as translation invariance. The aim of this study is the resolution to the problem of determination of the real properties of the system by providing a real formalism of the theory at all times.

In Section 2, the time evolution of the wavefunction is described by a equation with a real basis of solutions. An essential part of a calculation of the probabilities of different outcomes of experiments to occur is the superposition principle, which may be deduced from conservation of relative probabilities. As the constructive or destructive interference of wavefunctions affects the probabilities, it is necessary to establish this occurs also in a real formulation of the theory. The real formalism is then extended to quantum field theory in the third section. In particular, for scalar field theory, the expansion of the field, the commutation relations of the new annihilation and creation operators, the number operator and the Hamiltonian are real. By commuting the time-evolution operator of the scalar field with the Hamiltonian, time evolution of this field is given by a real operator equation. Finally, there exists a real form of the scattering matrix, which can be used to obtain directly finite amplitudes after a subtraction procedure.

2. A REAL FORMULATION OF QUANTUM MECHANICS

The wavefunction in quantum mechanics is generally complex and it is necessary to add wavefunctions according to the superposition principle to describe

a mixed state. While the reality of physical observables in quantum mechanics can be obtained through the hermiticity of the corresponding matrices, as the only measured quantities are the eigenvalues of these matrices, it would be useful to establish a formulation of quantum mechanics which is independent of complex numbers.

There exists a similarity transformation which maps a complex Hermitian matrix to a real matrix and an operation which transforms a complex operator equation to a real operator equation. For example the equation $i\frac{\partial\psi}{\partial t} = H\psi$ may be differentiated to give $\frac{\partial^2\psi}{\partial t^2} + H^2\psi = 0$. If the Hamiltonian is time-independent, this equation is equivalent to $\frac{\partial^2(T\psi)}{\partial t^2} + (THT^{-1})(T\psi) = 0$ where THT^{-1} is real. The real operator equation would have a real basis of solutions.

The superposition principle is a statement of the addition of eigenvectors of the operator representing the Hamiltonian. It can be deduced from the conservation of probabilities during the time evolution of a quantum system. Consider two states described by the wavefunctions ψ_1 and ψ_2 and a superposition of the two states $\psi_1 + \psi_2$. The sum of the evolved relative probabilities of these two states in the complex formalism is determined by

$$\int \psi_1(x,t')(\psi_1(x,t)^* + \psi_2(x,t)^*)\,d^3x + \int \psi_2(x,t')(\psi_1(x,t)^* + \psi_2(x,t)^*)\,d^3x \tag{2.1}$$

whereas the evolution of the sum of the relative probabilities is given by $\int(\psi_1 + \psi_2)(x,t')(\psi_1^* + \psi_2^*)(x,t)\,d^3x$. By the time-dependent Schrödinger equation,

$$\psi(x,t') = e^{-iH(t'-t)}\psi(x,t) \tag{2.2}$$

and the two expressions respectively are

$$\int e^{-iH(t'-t)}\psi_1(x,t)(\psi_1(x,t)^* + \psi_2(x,t)^*)\,d^3x \tag{2.3}$$

$$+ \int e^{-iH(t'-t)}\psi_2(x,t)(\psi_1(x,t)^* + \psi_2(x,t)^*)\,d^3x$$

$$\int e^{-iH(t'-t)}(\psi_1(x,t) + \psi_2(x,t))(\psi_1^* + \psi_2^*)(x,t)\,d^3x. \tag{2.4}$$

Conservation of the sum of the relative probabilities requires the equality of these two expressions or equivalently the linearity of the Hamiltonian operator. Linearity of the operator also implies that Leibnitz's property holds, which leads to an expression for the integral. It follows that if the wavefunction is analytic, the energy and other physical observables must be represented by linear differential operators and the superposition principle is valid.

For a real formulation of quantum mechanics to be valid, it is necessary that the superposition principle continues to hold. Nevertheless, after a similarity

transformation, it is possible to describe the mixed state by a linear combination of pure states represented by real wavefunctions with real coefficients. Since the linearity of the operators can be translated directly to higher-dimensional vector spaces, it is sufficient to express the solution of the second-order differential equation for $T\psi$ as a two-component vector

$$T\psi(x,t') = \begin{pmatrix} \sin(THT^{-1}(t'-t))(T\psi)(x,t) \\ \cos(THT^{-1}(t'-t))(T\psi)(x,t) \end{pmatrix}. \tag{2.5}$$

The wavefunction of the superposition of two states then would have the form

$$\begin{aligned} T(\psi_1+\psi_2)(x,t') &= \begin{pmatrix} \sin(THT^{-1}(t'-t))(T(\psi_1+\psi_2)(x,t) \\ \cos(THT^{-1}(t'-t))(T(\psi_1+\psi_2))(x,t) \end{pmatrix} \\ &= \begin{pmatrix} \sin(THT^{-1}(t'-t))(T\psi_1(x,t)+T\psi_2(x,t)) \\ \cos(THT^{-1}(t'-t))(T\psi_1(x,t)+T\psi_2(x,t)) \end{pmatrix} \\ &= T\psi_1(x,t')+T\psi_2(x,t'). \end{aligned} \tag{2.6}$$

It follows that constructive and destructive interference of the wavefunction occurs in the linear combination of the real-two component vectors, and the probabilities by evaluating the norms of the vectors after the summation of the wavefunctions.

In relativistic quantum mechanics, the four-momentum operator is $\hat{p}^\mu = i\hbar\frac{\partial}{\partial x_\mu}$. Replacing the imaginary unit i by $\left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$, this operator becomes $\left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)\hbar\frac{\partial}{\partial x_\mu}$, and the equation $\hat{p}^\mu\hat{p}_\mu\psi = m_0^2c^2\psi$ is

$$\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}\left(\Box + \frac{m_0^2c^2}{\hbar^2}\right)\psi = 0. \tag{2.7}$$

For a real wavefunction satisfying the Klein-Gordon equation, the charge density vanishes. However, it is possible, with the modifications in the real formalism that have been introduced, to represent charged particles. The complex current density (Greiner, 1999) equals

$$j_\mu = \frac{ie\hbar}{2m_0c^2}(\psi^*\nabla_\mu\psi - \psi\nabla_\mu\psi^*). \tag{2.8}$$

Since the solutions to the wave equations in a box of volume L^3 are

$$\psi = \sqrt{\frac{m_0c^2}{EL^3}}e^{-ip_\mu x^\mu} \tag{2.9}$$

the substitution of

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

for i yields

$$
\begin{aligned}
j_\mu = &\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \frac{e\hbar}{2m_0c^2} \frac{m_0c^2}{EL^3} \left[\left[\cos\left(\frac{1}{\hbar} p_\mu x^\mu\right) + \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \sin\left(\frac{1}{\hbar} p_\mu x^\mu\right) \right] \right. \\
&\times \frac{\partial}{\partial x^\mu} \left[\cos\left(\frac{1}{\hbar} p_\mu x^\mu\right) - \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \sin\left(\frac{1}{\hbar} p_\mu x^\mu\right) \right] \\
&- \left[\cos\left(\frac{1}{\hbar} p_\mu x^\mu\right) - \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \sin\left(\frac{1}{\hbar} p_\mu x^\mu\right) \right] \\
&\left. \times \frac{\partial}{\partial x^\mu} \left[\cos\left(\frac{1}{\hbar} p_\mu x^\mu\right) + \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \sin\left(\frac{1}{\hbar} p_\mu x^\mu\right) \right] \right] \\
= &\, e \frac{p_\mu}{EL^3} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \qquad (2.10)
\end{aligned}
$$

In particular, the charge is

$$
Q = \int d^3x j_0 = e \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \qquad (2.11)
$$

Interchange of ψ and ψ^* gives a charge of the opposite sign.

The postulates of quantum mechanics (Galindo and Pascual, 1990) also can be stated in the real formalism.

I. When a pure state of the quantum system is represented by a unit ray in a Hilbert space, the conjugate is replaced by transposition. This substitution includes both the complex and Hermitian conjugates for state vectors, since the two transformations are equivalent on this wavefunctions. When the wavefunction has several components, it will be an element in the tensor product of two matrix spaces, and transposition must be applied to both spaces.

II. In the representation of observables by Hermitian matrices, a real formulation is achieved by similarity transformations which render the matrix real. The similarity transformations exist in all finite-dimensional spaces. This property can be extended to infinite dimensions

III. Given that the probability of obtaining a value λ associated with a state $|\psi\rangle$ from a Borel set $\Delta \subset \mathbb{R}$ upon measurement of an observable A is

$$
P_{A,\psi}(\Delta) = \| E_A(\Delta) \|^2 \qquad (2.12)
$$

where $E_A = |\psi\rangle_{A|_\Delta A|_\Delta}\langle\psi|$, the probability also can be expressed as the squared norm of a two-dimensional vector in the real formalism.

IV. If the spectrum of A is a subset of a set Δ, and the quantum system is described by the density matrix ρ, then a measurement of A will be detemined by the density matrix

$$\rho_{A,\Delta} = \frac{1}{\mathrm{Tr}[\rho E_A(\Delta)]} \sum_{\alpha\in\Delta} E_{M_\alpha} \rho E_{M_\alpha}. \tag{2.13}$$

The density matrix, which has the form $\rho = \sum_i |\psi_i\rangle p_i \langle\psi_i|$, satisfies the relations $\rho^\dagger = \rho$, $\rho \geq 0$ and $\rho \geq \rho^2$. The Hermitian condition becomes $\rho^T = \rho$ in the real formalism. The last inequality is again satisfied since

$$\begin{aligned}
\int d^3x \psi^T \rho^2 \psi &= \sum_{i,j} p_i p_j \int d^3x \psi^T \psi_i \psi_i^T \psi_j \psi_j^T \psi \\
&\geq \sum_{i,j} p_i p_j \int d^3x |\psi^T \psi_i \psi_j^T \psi| \\
&= \int d^3x \left[\sum_i p_i |\langle\psi_i|\psi\rangle| \right]^2 \\
&\leq \sum_i p_i \int d^3x |\psi_i^T \psi|^2 = \int d^3x \psi^T \rho \psi.
\end{aligned} \tag{2.14}$$

V. The time-evolution of the wavefunction and density matrix is given by

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = H(t)|\psi(t)\rangle \tag{2.15}$$

and

$$i\hbar \frac{d\rho(t)}{dt} = [h(t), \rho(t)]. \tag{2.16}$$

These equations would be replaced by

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \hbar \frac{d}{dt} |\psi(t)\rangle = H(t)|\psi(t)\rangle \tag{2.17}$$

and

$$-\hbar^2 \frac{d^2\rho}{dt^2} = [H(t), [H(t), \rho(t)]]. \tag{2.18}$$

VI. The canonical commutation relations are

$$\begin{aligned}
&[X_i, P_j] = 0 \\
&[P_i, P_j] = 0 \qquad\qquad i, j = 1, \ldots, N. \\
&[X_i, P_j] = i\hbar\delta_{ij}
\end{aligned} \tag{2.19}$$

The real form of the commutator of the position and momentum operators would be

$$[X_i, P_j] = \hbar \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \delta_{ij}. \tag{2.20}$$

It is also known that the imaginary unit always occurs with $\hbar$ in quantum mechanics, implying that the classical limit $\hbar \to 0$ is always real. It is sufficient therefore to separate these two quantities in the real description of quantum effects. Amongst the methods for eliminating the dependence on the imaginary unit are the squaring of the absolute values of the variables and the substitution of i by the real antisymmetric matrix.

An example is the uncertainty principle $\Delta x \Delta p \geq \frac{\hbar}{2}$, which is derived from $[x, p] = i\hbar$ and an inequality for the square absolute value $|\langle\psi|(x - \langle x\rangle)(p - \langle p\rangle)\psi|^2$. Alternatively, from Eq. (2.20), $\Delta X_i \Delta P_j \geq \frac{\hbar}{2}\delta_{ij}\left(\begin{smallmatrix} I & 0 \\ 0 & I \end{smallmatrix}\right)$, where the product of the expectation values includes the matrix product $\left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)^{\mathrm{T}}\left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$.

When the wavefunction $\psi(x, t) = Ne^{i/\hbar S(x.t)}$ solves the Schrödinger equation (Galindo and Pascual, 1990), $i\hbar\frac{\partial\psi(\vec{x},t)}{\partial t} = [-\frac{\hbar^2}{2m}\nabla^2 + V(\vec{x})]\psi(\vec{x}, t)$,

$$\frac{\partial S}{\partial t} + \frac{1}{2m}(\vec{\nabla}S)\cdot(\vec{\nabla}S) + V - \frac{i\hbar}{2m}\nabla^2 S = 0 \tag{2.21}$$

with the Hamilton–Jacobi equation arising in the $\hbar \to 0$ limit. Both the equation and the solution, given by $S = S_0 - i\hbar S_1 + (-i\hbar)^2 S_2 + \cdots$, also have the factor of i combined with $\hbar$.

The imaginary unit may be eliminated either through the replacement of Eq. (2.21) by

$$\frac{\partial S}{\partial t} + \frac{1}{2m}(\vec{\nabla}S)^2 + V - \frac{\hbar}{2m}\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\nabla^2 S = 0 \tag{2.22}$$

where the solution is the product of an analytic function and a 2×2 matrix, or with real equations for the position and momentum variables using Hepp's procedure (Hepp, 1974). Let $X_{\bar{\hbar}} = \hbar^{\frac{1}{2}}\bar{X}$ and $P_{\bar{\hbar}} = \hbar h^{1/2}\bar{P}$, and consider the matrix

$$W(\xi, \pi) = \exp\left[\frac{i}{\hbar}(\pi X_{\bar{\hbar}} - \xi P_{\bar{\hbar}})\right] \tag{2.23}$$

which may be expressed equivalently as

$$W(\xi, \pi) = \cos\left[\frac{1}{\hbar}(\pi X_{\bar{\hbar}} - \xi P_{\bar{\hbar}})\right] + \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\sin\left[\frac{1}{\hbar}(\pi X_{\bar{\hbar}} - \xi P_{\bar{\hbar}})\right]. \tag{2.24}$$

It has been proven that

$$W^{\dagger}(x_{\rm cl}(0),\, p_{\rm cl}(0))X_{\hbar}(t)W(x_{\rm cl}(0),\, p_{\rm cl}(0)) \underset{\hbar \to 0}{\sim} x_{\rm cl}(t) + \hbar^{1/2}\bar{X}_{\rm osc}(t)$$
$$W^{\dagger}(x_{\rm cl}(0),\, p_{\rm cl}(0))P_{\hbar}(t)W(x_{\rm cl}(0),\, p_{\rm cl}(0)) \underset{\hbar \to 0}{\sim} p_{\rm cl}(t) + \hbar^{1/2}\bar{P}_{\rm osc}(t) \quad (2.25)$$

where $(x_{\rm cl}(t),\, p_{\rm cl}(t))$ is the classical trajectory in phase space, $\frac{d\bar{X}_{\rm osc}(t)}{dt} = \frac{1}{m}\bar{P}_{\rm osc}(t)$ and $\frac{d\bar{P}_{\rm osc}(t)}{dt} = -V''(x_{\rm cl}(t))\bar{X}_{\rm osc}(t)$, with the initial conditions being $\bar{X}_{\rm osc}(0) = X$ and $\bar{P}_{\rm osc}(0) = P$ (Galindo and Pascual, 1990; Hepp, 1974).

By Wigner's theorem, the action of any symmetry transformation on a Hilbert space is a one-to-one linear or anti-linear isometry (Galindo and Pascual, 1990; Wigner, 1959). For an anti-unitary transformation $\langle\psi_G|\phi_G\rangle = \langle\psi|\phi\rangle^*$ in the complex formalism. The transposition is the only operation on real matrices other than the identity which preserves the orthogonality relation $O^{\rm T}O = I$ and which is an involution. The existence of the involution is necessary for the new state to be defined symmetrically with respect to a real origin in state space. It can be deduced in the real formalism that this is the only form of a symmetry transformation other than the unitary mapping, and therefore the possibility of another complex phase arising in the transformation is eliminated.

3. QUANTUM FIELD THEORY

A quantum scalar field can be expanded in terms of sine and cosine functions,

$$\phi(x) = \frac{1}{(2\pi)^3}\int \frac{d^3\vec{p}}{2p^0}[a_c(\vec{p})\cos(p\cdot x) + a_s(\vec{p})\sin(p\cdot x)]. \quad (3.1)$$

When the coefficients $a_c(p)$ and $a_s(p)$ are elevated to operators, the commutation relations are

$$[a_c(\vec{p}), a_s(\vec{p}')] = -i(2\pi)^3 p^0\delta(\vec{p} - \vec{p}')$$
$$[a_c(\vec{p}), a_c(\vec{p}')] = 0$$
$$[a_s(\vec{p}), a_s(\vec{p}')] = 0. \quad (3.2)$$

The equal-time commutation relations for ϕ and the conjugate momentum Π have been postulated in analogy with $[q,\, p] = i\hbar\mathbf{1}$. As the uncertainty relation in quantum mechanics is $\Delta q\,\Delta p \geq \frac{\hbar}{2}$, it follows that $\Delta\phi(\vec{x}, t)\Delta\Pi(\vec{x}', t) \geq \frac{\hbar}{2}\delta(\vec{x} - \vec{x}')$. The number operator would be

$$N = \frac{1}{(2\pi)^3}\int \frac{d^3\vec{p}}{2p^0}a^{\dagger}(\vec{p})a(\vec{p})$$
$$= \frac{1}{(2\pi)^3}\int \frac{d^3\vec{p}}{2p^0}[a_c(\vec{p}) + ia_s(\vec{p})][a_c(\vec{p}) - ia_s(\vec{p})]$$

$$= \frac{1}{(2\pi)^3}\int \frac{d^3\vec{p}}{2p^0}[(a_c(\vec{p})^2 + a_s(\vec{p})^2) + i[a_s(\vec{p}), a_c(\vec{p})]]$$

$$= \frac{1}{(2\pi)^3}\int \frac{d^3\vec{p}}{2p^0}\left\{(a_c(\vec{p})^2 + a_s(\vec{p})^2) - \frac{1}{2}\right\} \tag{3.3}$$

and Hamiltonian is

$$H = \frac{1}{2}\frac{1}{(2\pi)^3}\int \frac{d^3\vec{p}}{2p^0}p^0(a(\vec{p})a^\dagger(\vec{p}) + a^\dagger(\vec{p})a(\vec{p}))$$

$$= \frac{1}{2}\frac{1}{(2\pi)^3}\int \frac{d^3\vec{p}}{2p^0}p^0[[a_c(\vec{p}) + ia_c(\vec{p})][a_c(\vec{p} + ia_c(\vec{p})]$$

$$+ [a_c(\vec{p}) - ia_c(\vec{p})][a_c(\vec{p}) + ia_s(\vec{p})]]$$

$$= \frac{1}{(2\pi)^3}\int \frac{d^3\vec{p}}{2p^0}p^0(a_c(\vec{p})^2 + a_s(\vec{p})^2). \tag{3.4}$$

Although equal-time commutation relations have a factor of i, $[\phi(\vec{x}, t), \Pi(\vec{x}', t)] = i\hbar\delta(\vec{x} - \vec{x}')$, leading to the equation $i\hbar\dot{\phi} = [\phi, H]$, the latter relation can be replaced by

$$[i\hbar\dot{\phi}, H] = i\hbar\frac{d}{dt}[\phi, H] = i\hbar\frac{d}{dt}(i\hbar\dot{\phi}) = -\hbar^2\ddot{\phi} = [\phi, [\phi, H]]. \tag{3.5}$$

In the Dirac picture, both the wavefunction and the operators are time-dependent,

$$\psi(t) = U(t, t_0)\psi(t_0)$$

$$i\frac{\partial U(t, t_0)}{\partial t} = H_I U(t, t_0) \tag{3.6}$$

with solution $U(t, t_0) = T\exp(-i\int_{t_0}^{t} dt' H_I) \equiv T\exp(-i\int_{t_0}^{t} dt' V(t'))$. The S-matrix then would be $S = U(\infty, -\infty)$. However, Eq. (3.6) can be differentiated to give

$$\frac{\partial^2 U(t, t_0)}{\partial t^2} + H_I^2 U(t, t_0) = 0 \tag{3.7}$$

and $U(t, t_0) = c_1 T\cos(\int_{t_0}^{t} dt' V(t')) + c_2 T\sin(\int_{t_0}^{t} dt' V(t'))$. Instead of an S-matrix, another operator can be defined

$$\hat{S} = c_1 T\cos\left(\int_{-\infty}^{\infty} dt V(t)\right) + c_2 T\sin\left(\int_{-\infty}^{\infty} dt V(t)\right)$$

$$= c_1 T\cos\left(\int d^4x L_I\right) - c_2 T\sin\left(\int d^4x L_I\right). \tag{3.8}$$

In the standard renormalization procedure, counterterms are introduced to cancel the divergences in the perturbative expansion of the S-matrix. The infinities

arising from the coincidence of the field operators do not have the same effect on the $\hat{S}$-matrix, since the trigonometric functions do not diverge. Indeed, consistency of the $\hat{S}$-matrix formalism follows if the finite part is separated either in the operator expansion of L_I in Eq. (3.8) or the series expansion of $\hat{S}$, and divergent part is subtracted, while other possible finite renormalizations may be included.

$$\hat{S}_{\text{ren}} = c_1 T \cos\left(\int d^4x L_I - \int d^4x L_{I\,\text{div}}\right) - c_2 T \sin\left(\int d^4x L_I - \int d^4x L_{I\,\text{div}}\right). \tag{3.9}$$

The two components of $\hat{S}$ represent the $t \to \infty$, $t_0 \to -\infty$ limits of orthogonal solutions to Eq. (3.10). If $\hat{S}_{\text{ren}}$ is regarded as a two-component column vector

$$\hat{S}_{\text{ren}} = \begin{pmatrix} c_1 T \cos\left(\int d^4x L_I\right)_{\text{ren}} \\ -c_2 T \sin\left(\int d^4x L_I\right)_{\text{ren}} \end{pmatrix} \tag{3.10}$$

$$\langle 0|\hat{S}^{\dagger}_{\text{ren}}\hat{S}_{\text{ren}}|0\rangle = |c_1|^2\langle 0|\left(T\cos\int d^4x L_I\right)^2_{\text{ren}}|0\rangle + |c_2|^2\langle 0|\left(T\sin\int d^4x L_I\right)^2_{\text{ren}}|0\rangle. \tag{3.11}$$

When $|c_1|^2 = |c_2|^2 = 1$, this inner product can be set equal to 1. For a real $\hat{S}_{\text{ren}}$ matrix, c_1, c_2 are real, and the condition on these coefficients is $c_1^2 = c_2^2 = 1$.

Correlation functions then could be defined to be

$$\begin{aligned}\hat{G}(x_1, \ldots x_n) &= \frac{{}_D\langle 0|T\phi_D(x_1)\ldots\phi_D(x_n)\hat{S}_{\text{ren}}|0\rangle_D}{{}_D\langle 0|\hat{S}_{\text{ren}}|\rangle_D} \\ &= \frac{D\langle 0|T\phi_D(x_1)\ldots\phi_D(x_n)\left[c_1 T\cos\left(\int L_I d^4x\right)_{\text{ren}} - c_2 T\sin\left(\int L_I d^4x\right)_{\text{ren}}\right]|0\rangle_D}{D\langle 0|\left[c_1 T\cos\left(\int L_I d^4x\right) - c_2 T\sin\left(\int L_I d^4x\right)\right]}.\end{aligned} \tag{3.12}$$

The Green functions are also conventionally derived as varational derivatives of the partition function $Z[J]$. This partition function is real in the Euclidean formalism, with

$$\begin{aligned} Z_E[J] &= \frac{\Phi_E[J]}{\Phi_E[0]} \\ \Phi_E[J] &= \int D\phi e^{-\int d^4x(L+J\phi)}. \end{aligned} \tag{3.13}$$

The use of the Wick rotation to Euclidean space is also known to be necessary in the renormalization of the electric charge, where the vertex graph receives

higher-loop corrections. At one loop, $e\bar{u}\gamma_\lambda u \to e\bar{u}(\gamma_\lambda + \Gamma_\lambda)u$, where

$$\Gamma_\lambda = -ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{\gamma_\mu(\gamma_\sigma(p^\sigma + k^\sigma) + m)\gamma_\lambda(\gamma_\rho(p^\rho + k^\rho) + m)\gamma^\mu}{[(p' - k)^2 - m^2 + i\epsilon][(p - k)^2 - m^2 + i\epsilon][k^2 + i\epsilon]}. \tag{3.14}$$

To establish that the coefficient $F_1(q^2), q = p' - p$ of γ_λ in Γ_Λ is real, a Wick rotation to Euclidean momentum is required, as it introduces an extra factor of i.

The coupling of the electron to the electromagnetic field leads to the introduction of a covariant derivative. Although it is customary to include an imaginary unit in the covariant derivative, it is sufficient to use $D_\mu = \partial_\mu + eA_\mu$, as the vector $D_\mu\phi$, with ϕ a scalar field, transforms covariantly when $A_\mu \to g^{-1}\partial_\mu g + g^{-1}A_\mu g$. Based on this covariant derivative, the operator in the Pauli equation

$$\left(D^\mu D_\mu + \frac{1}{4}e[\gamma^\mu, \gamma^\nu][D_\mu, D_\nu] + m^2\right)\psi = 0 \tag{3.15}$$

is clearly real except for the term containing $[\gamma^\mu, \gamma^\nu]$. In the Majorana representation (Itzykson and Zuber, 1980), γ^i is purely imaginary and the commutator $[\gamma^i, \gamma^j]$ would be purely real. Since $[D_\mu, D_\nu]\psi \to -eF_{\mu\nu}\psi$, the operator $\frac{1}{4}[\gamma^i, \gamma^j][D_i, D_j]$ is real in this representation. Moreover, the extra term yields a magnetic moment term in the nonrelativistic limit, and it is consistent with the reality of the other terms.

The real formulation of the Dirac equation $(i\gamma^\mu\partial_\mu - m)\Psi = 0$ is

$$\begin{aligned} \gamma^\mu\partial_\mu\psi\gamma_2\gamma_1\gamma_0 - m\psi &= 0 \\ \Psi &= \psi u_1 \\ u_1 &= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \end{aligned} \tag{3.16}$$

where the imaginary unit is interpreted to be the generator of rotations in the spin plane and ψ is a multivector (Hestenes, 1967, 1975). The action of the parity, charge conjugation and time reversal operators can be defined on the wavefunction ψ (Hestenes, 1967).

A real form of a quantum field theory can be derived from the identification of the imaginary unit with

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Felsager (1998), which is the negative of the matrix considered in Section 2. Similarly, another real form of the Dirac equation can be defined with the identification of the imaginary unit i with

$$\begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix},$$

where I is a 2×2 identity matrix. When a complex scalar field is coupled to an electromagnetic field, an extra term arises in the current and there is a nonlinear term in the equation of motion. This problem can be circumvented through the Duffin-Kemmer-Petiau equation (Petiau, 2006; Kemmer, 1938, 1939; Duffin, 1938; Dreschler and Mayer, 1977)

$$(i\beta^k \partial_k - m)u = 0 \tag{3.17}$$

where $\{\beta_k\}$ are 5×5 real matrices. The identification of the imaginary unit with an antisymmetric matrix is valid only if the order of the matrix is even. If $\{\beta_k\}$ is trivially extended to a 6×6 matrix, and the spinor u is replaced by a six-component spinor, this equation can be cast in entirely real form with the identification of i with $\left(\begin{smallmatrix} 0 & -I_3 \\ I_3 & 0 \end{smallmatrix}\right)$

The quantization of the Dirac action can be extended to perturbation theory by using the inverse of the differential operator for the propagator. For the first formulation, it would be

$$\left[-\gamma^\mu p_\mu \gamma_2 \gamma_1 \gamma_0 - m \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \right]^{-1} \tag{3.18}$$

given that a Laplace transform is used to derive the momentum space rules, while in the second formulation, it is

$$\left[-\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \gamma^\mu p_\mu - m \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \right]^{-1}. \tag{3.19}$$

These expressions can be fully evaluated in a series expansion with terms containing matrix products and would be included in the calculations of perturbation theory.

4. CONCLUSION

The formulation of physical models in terms of complex variables leads to the problem of the reality of quantities in the theory. In quantum mechanics, one of the physical consequences of the complex numbers is the superposition of complex wavefunctions to describe a mixed state. However, with the diagonalization of the Hermitian matrices representing the maximal set of simultaneously measurable physical observables, it becomes a real superposition of orthogonal states described

by real wavefunctions. While these wavefunctions would have two components for the sine and cosine functions, further work may lead to a real formalism of quantum theory which is not based strictly on the isomorphism between $\mathbb{C}$ and $\mathbb{R}^2$. For example, in the Appendix, it is shown that the category of complex numbers can be defined in terms of a function of the real numbers and that it may be done without resorting to multi-component vectors. In quantum field theory, the field can be expanded in terms of sine and cosine functions. This yields a formula for the $\hat{S}$-matrix and for the correlation functions involving trigonometric functions. The finiteness of the trigonometric function with infinite argument provides a regularization of short-distance divergences and implies that divergences in the $\hat{S}$-matrix can be removed in the argument.

APPENDIX

The category of complex numbers can be defined by a condition which can be imposed on real numbers and not vectors with more than one component. It will be demonstrated also that there is a class of functions of a complex variable which satisfies the properties of a functor, distinguishing between complex and real numbers. These results provide a link between a complex and an intrinsically real formulation of quantum mechanics.

Consider three sets,

$$\begin{aligned} C_1 &= \{z|z^2 + |z^2| = 0\} \\ C_2 &= \{z|2|z|^2 > z^2 + |z^2| > 0\} \\ C_3 &= \{z|z^2 + |z^2| = 2|z|^2\}. \end{aligned} \tag{A.1}$$

Then $C = C_1 \cup C_2 \cup C_3$ is a category such that there exist mappings

$$\begin{aligned} \phi_1 &: C_1 \times C_1 \to C_3 \\ \phi_2 &: C_1 \times C_2 \to C_2 \\ \phi_3 &: C_1 \times C_3 \to C_1 \\ \phi_4 &: C_2 \times C_2 \to C_1 \cup C_2 \cup C_3 \\ \phi_5 &: C_2 \times C_3 C_2 \\ \phi_6 &: C_3 \times C_3 \to C_3. \end{aligned} \tag{A.2}$$

The set of mappings $\{\phi_1\} \cup \ldots \cup \{\phi_6\}$ are functors.

An ordered set based on the real numbers can be regarded as a category (Pareigis, 1970). The set defined by the ordered pairs (a, b) with the morphism given by $\eta(a, b) = (a, b)$ if $a \leq b$ and $\eta(a, b) = 0$ if $a \geq b$ is isomorphic to half of the complex plane. Consider also the set defined by the ordered pairs (a, b), with

the morphism $\tilde{\eta}(a, b) = (a, b)$ if $a \geq b$ and ϕ otherwise. The union of the two sets together with both types of morphisms yields the category of complex numbers.

The geometrical description of the complex numbers by a function of the real numbers follows from the derivation of the properties of a function of a complex variable from the characteristics of a real function and the compactification of the real axis to a circle with $-\infty$ and ∞ identified.

The transition from the complex to the real variable may be deduced from the equality between the imaginary unit i and $\sqrt{\frac{1/0^+}{1/0^-}}$ or equivalently the square root of the limit of the ratio of x tending to zero in opposite directions. Regarding the limit as a map from a point in the neighbourhood of zero to zero, the square root of the limit would be a map that point in a neighbourhood to another point. Since the square root map must be performed twice to reach the origin, it must be defined such that both square root limits tend towards the third. This would imply the existence of a limit point of the square root map which does not belong to the real numbers. By L'Hopital's rule, the square root limit of the ratio of derivatives of two function would be equal to the square root limit of the ratio of the functions. Based on the ratio to a standard function, all of the derivatives can be defined at i and the function in a neighbourhood of i. The overlapping of neighbourhoods can be used to define the function on a domain in the complex plane.

Consider, for example, the function

$$F(z) = z + \frac{1}{z} + \frac{1}{z + \frac{1}{z}} + \frac{1}{z + \frac{1}{z} + \frac{1}{z + \frac{1}{z}}} + \cdots. \tag{A.3}$$

Since the sequence $\{F_n\}$ defined by the truncation of the series at the nth term satisfies the recursion relation $F_{n+1} = F_n + \frac{1}{F_n}$ and the series diverges for real values of $z \geq 1$. The series also diverges for positive values of z between 0 and 1, since $F(1/z) = F(z)$. For negative values of z, the function equals $-\infty$ and it is infinite for all $z \in R$.

If $z = \alpha i$,

$$\begin{aligned} F(\alpha i) &= \alpha i + \frac{1}{\alpha i} + \frac{1}{\alpha i + \frac{1}{\alpha i}} + \cdots \\ &= i\left(\alpha - \frac{1}{\alpha}\right) + \frac{1}{i\left(\alpha - \frac{1}{\alpha}\right)} + \frac{1}{i\left(\alpha - \frac{1}{\alpha} - \frac{1}{\alpha - \frac{1}{\alpha}}\right)} + \cdots \end{aligned} \tag{A.4}$$

and $F(\alpha i)$ has poles when

$$\alpha - \frac{1}{\alpha} = 0$$

$$\alpha - \frac{1}{\alpha} - \frac{1}{\alpha - \frac{1}{\alpha}} = 0$$

$$\alpha - \frac{1}{\alpha} - \frac{1}{\alpha - \frac{1}{\alpha}} - \frac{1}{\alpha - \frac{1}{\alpha} - \frac{1}{\alpha - \frac{1}{\alpha}}} = 0$$

$$\vdots \tag{A.5}$$

with solutions

$$\pm 1$$

$$\frac{1 \pm \sqrt{5}}{2}, \frac{-1 \pm \sqrt{5}}{2}$$

$$\frac{1}{4}\left[1 + \sqrt{5} \pm \sqrt{7 + \frac{\sqrt{5}}{2}}\right], \frac{1}{4}\left[1 - \sqrt{5} \pm \sqrt{7 - \frac{\sqrt{5}}{2}}\right] \tag{A.6}$$

$$\vdots \tag{A.7}$$

Although the terms at finite order for α located away from these poles are finite, they increase in magnitude since $(((\alpha - \frac{1}{\alpha})^2 - 1)^2 - 1)^2 \ldots$ converges to zero if $\alpha - \frac{1}{\alpha} < \frac{1+\sqrt{5}}{2}$ and ∞ if $\alpha - \frac{1}{\alpha} > \frac{1+\sqrt{5}}{2}$.

If the circle is the cut and continued to form a spiral, this would provide a representation of the values of the function on real axis, beginning with the first point representing ∞, given that $F(1) = \frac{1}{0^+}$, $F(-1) = \frac{1}{0^-}$. A segment of the spiral then can be mapped to the circle $|z| = 1$, yielding a geometrical description of complex numbers of unit magnitude. Complex numbers of magnitude n then may be mapped bijectively onto the segment of the spiral with endpoints $F(2n - 3)$ and $F(2n - 1)$. The complex numbers with rational magnitude would equal fractions of the the complex numbers with integer magnitude. The remaining complex numbers would be derived through the method of Cauchy completion of $\mathbb{Q}$ into $\mathbb{R}$.

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